raised to the n'th power.[6] This process, however, involves more advanced concepts of the matrix calculus. Also, the solution in terms of hyperbolic functions can be obtained from the finite difference equation (5) by conventional methods, but requires more algebraic manipulation in that case.

CONCLUSIONS

It may be concluded that methods for the solution of the given problem are available which take nothing for granted but Ohm's and Kirchhoff's laws, and yet arrive at the goal with no more, or even less, algebraic manipulation than the "postulatory" method in the paper under discussion. Two of the possible methods have been presented in this note.

The finite difference method leads to the solution almost immediately. Its fundamental recursion formula (5) follows directly from the basic relation $\sum i = 0$ at a network junction, which can be formulated by inspection. A few more steps lead to explicit expressions for the coefficients $a_{m,n}$ and $b_{m,n}$, although, in general, solution by actual recursion according to (5) would be adequate. The second method, using image parameters suitable for cascade connection of symmetrical structures, may be even more attractive to the communications engineer.

[6] The first method is closely related, in mathematical terms, to a similarity transformation which equalizes the elements in the principal diagonal.

The Measurement of Antenna Impedence Using a Receiving Antenna*

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***Summary*—Energy from a remote transmitter excites a receiving antenna that is erected vertically over a large conducting plane and base-loaded by a vertical slotted coaxial cavity of variable length. From measurements of the location and half-power width of resonance curves, the combined phase and damping factors for the two ends of the cavity are determined. By measuring these factors for the lower end of the cavity separately, those of the upper end are determined and used to calculate the impedance of the antenna. In effect, the receiving antenna is the generator driving the coaxial line, and it is the impedance of this generator that is measured. Curves of the measured impedance as a function of the electrical length of the antenna are given. Excellent agreement is obtained between impedances measured in this manner for the receiving antenna and corresponding impedances of the same antenna when base-driven through the slotted section. Both sets of measurements are in good agreement with the King-Middleton second-order theory.**

IS THE IMPEDANCE of an antenna that is loaded and used for reception equal to the impedance of the same antenna when used for transmission with the load replaced by a generator? The answer is simple. Yes, the impedances necessarily are equal if they are *defined to be the same*, and this is both useful and conventional. This is done in terms of an "equivalent" series circuit in which the receiving antenna is replaced by a concentrated emf V and a lumped internal impedance Z_0 in series with the lumped load Z_L. This circuit is rigorously equivalent to the antenna with load for the current I_0 entering and leaving the load, as is readily established using Thévenin's Theorem.[1] In the symmetrical, center-loaded antenna in Fig. 1(a) the open circuit voltage across AB as in Fig. 1(b), due to the action of the electric field, is $V = V_{AB}$ (open). Thévenin's Theorem states that I_0 in Fig. 1(a) is the same as I_0 in Fig. 1(c) if $V = V_{AB}$ (open) and Z_0 is the impedance looking into the terminals AB in Fig. 1(b) with the electric field E equal to zero. Hence,

$$I_0 = V/(Z_0 + Z_L). \tag{1}$$

Note that by definition Z_0 is the impedance of the antenna *as if driven by a potential difference across its terminals*. Hence, it is identically the transmitting impedance.

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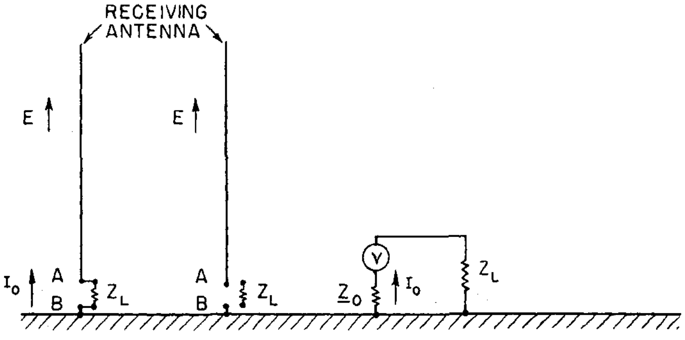
[1] Cruft Electronics Staff, "Electronic Circuits and Tubes," McGraw-Hill Book Co., New York, N. Y., p. 110; 1947.

V in (1) is that concentrated voltage which would maintain the same current I_0 in the load Z_L when connected in series with the lumped impedance Z_0 of the antenna as exists by action of the electric field along the loaded antenna. $V = V_{AB}$ (open) is defined in the literature[2,3] in terms of the electric field E and the dimensions and orientation of the antenna. Note that the simple circuit in Fig. 1(c) is equivalent to the actual antenna *only* in the determination of I_0. If the voltage V is connected in series with Z_L and the antenna, I_0 is the same as in the receiving antenna if V is properly defined, but I_z elsewhere on the antenna is not.



(a) (b) (c)

Fig. 1—Equivalent circuit for I_0 in receiving antenna.

The impedance of the receiving antenna, defined as Z_0 in (1), may be measured with the apparatus shown in Fig. 2. Z_L is the input impedance of a vertical section of slotted line terminated in a sensitive detector and a piston, and the antenna is the vertical extension of the inner conductor above a large ground screen. V is the equivalent concentrated generator driving the line, and Z_0 is the internal impedance of the generator, i.e., the impedance of the antenna. Thus, the experimental problem is to measure the internal impedance Z_0 of the generator driving the line. This can *not* be done using either the conventional standing-wave-ratio method[4] or the distribution-curve method[4] since these involve only the load impedance. However, Z_0 is readily determined using the resonance-curve method.[4,5] It is necessary merely to determine the position and half-power width of the resonance curve obtained by moving the piston terminating the line for each length h of the antenna.

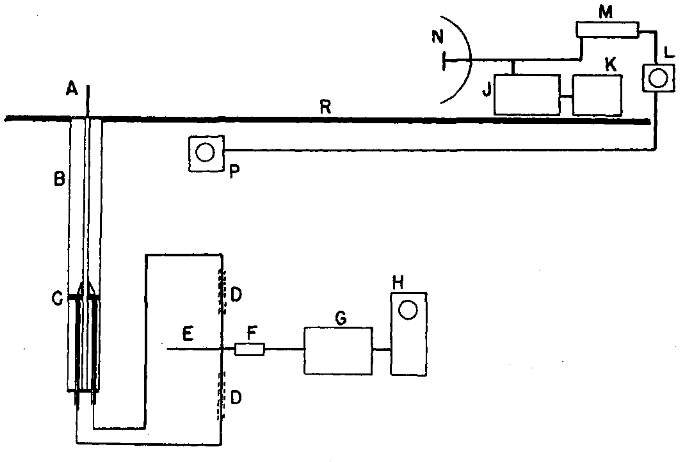


Fig. 2—Schematic diagram of complete equipment.

A = receiving antenna
B = measuring line
C = movable piston
D = double line stretcher
E = shunt stub
F = bolometer mount
G = tuned bolometer amplifier
H = Ballantine voltmeter
J = transmitter
K = modulator
L = wavemeter crystal current
M = wavemeter
N = transmitting antenna
P = remote crystal current
R = ground plane.

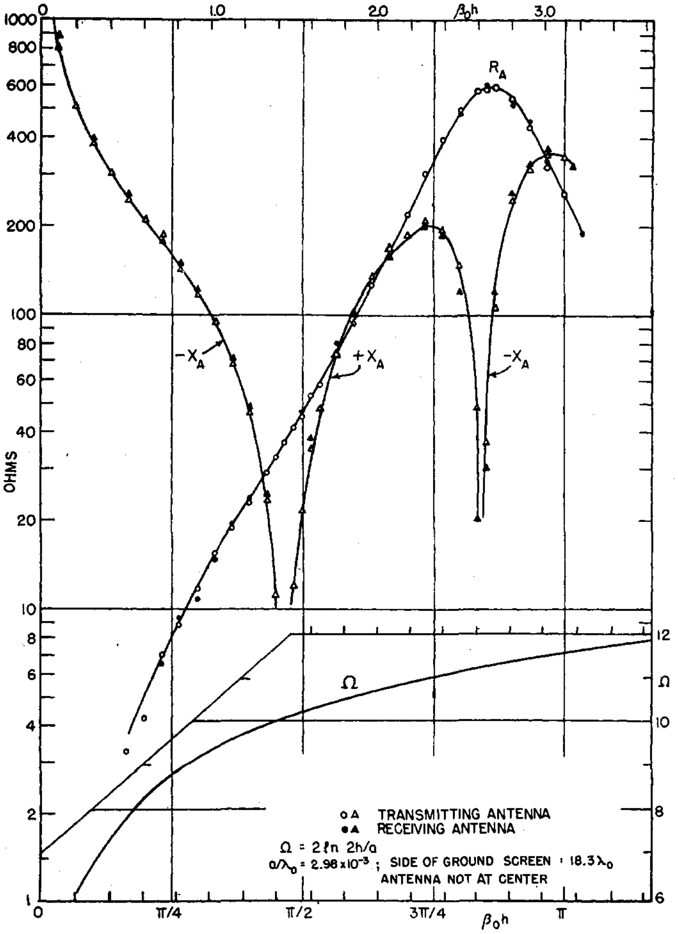


Fig. 3—Impedance of antenna by reception and transmission methods.

The resonance-curve method is expressed concisely using the terminal functions ρ_o and Φ_o of Z_0 at $z=0$, and ρ_s and Φ_s of Z_s at the end $z=s$ of the coaxial line with characteristic impedance $Z_c \doteq R_c$. These functions are defined by

$$Z_0 = Z_c \coth(\rho_o + j\Phi_o); \; Z_s = Z_c \coth(\rho_s + j\Phi_s). \quad (2)$$

The over-all phase function $(\beta s + \Phi_o + \Phi_s)$ is determined directly from the resonant length of the line; the over-all attenuation function $(\alpha s + \rho_o + \rho_s)$ from the half-power width, using

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[2] R. King and C. W. Harrison, Jr., "The receiving antenna," Proc. I.R.E., vol. 32, pp. 18–49; January, 1944.

[3] R. W. P. King, H. R. Mimno, and A. H. Wing, "Transmission Lines, Antennas, and Wave Guides," McGraw-Hill Book Co., New York, N. Y., p. 164; 1945.

[4] D. D. King, "Impedance measurements on transmission lines," Proc. I.R.E., vol. 35, pp. 507–514; May, 1947.

[5] R. King, "Transmission-line theory and its application," *Jour Appl. Phys.*, vol. 14, pp. 577–600; November, 1943.

$$B = \beta s_n + \Phi_0 + \Phi_s = n\pi; \quad n = 1, 2, \cdots, \tag{3}$$

$$A = \alpha s_n + \rho_o + \rho_s = \beta W/2, \tag{4}$$

where α and $\beta = 2\pi/\lambda$ are, respectively, the attenuation and phase constants of the line, W is the full width of the resonance curve, and s_n is the length of the line at resonance. By predetermining Φ_s and ρ_s for the piston with its two detecting loops, measuring s directly, obtaining β from the measured wavelength, and computing the small quantity α from the dimensions and material of the line, ρ_o and Φ_o may be evaluated from (3) and (4).

$Z_0 = R_0 + jX_0$ is determined from curves of ρ_o and Φ_o, using

$$Z_0 = R_0 + jX_0 = \frac{R_c[\sinh 2\rho_o - j \sin 2\Phi_o]}{\cosh 2\rho_o - \cos 2\Phi_o}. \tag{5}$$

Since ρ_o and Φ_o are much more slowly varying than R_o and X_o, it is more accurate to draw smooth curves of ρ_o and Φ_o through the experimental points and use these to determine R_o and X_o than to substitute the experimental values of ρ_o and Φ_o directly in (5). Experimental curves of R_o and X_o are in Fig. 3. In Figs. 4 and 5 experimental points are compared with theoretical curves of the King-Middleton second-order theory.[6]

The vertical slotted line used in the measurements is that described by D. D. King.[7] It is provided with a long taper so that the gap may be kept sufficiently small to make terminal effects negligible. Styrofoam was used for the single insulator at the upper end of the line. The ground screen was a 36-foot square of sheet aluminum. The operating frequency was 500 mc.

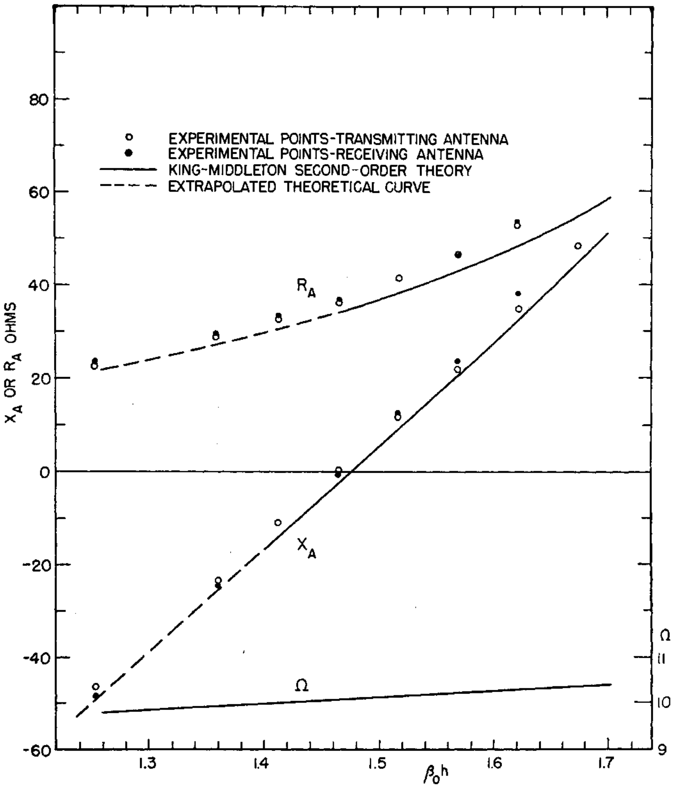

Fig. 4—Theoretical impedance near resonance with experimental points from Fig. 3.

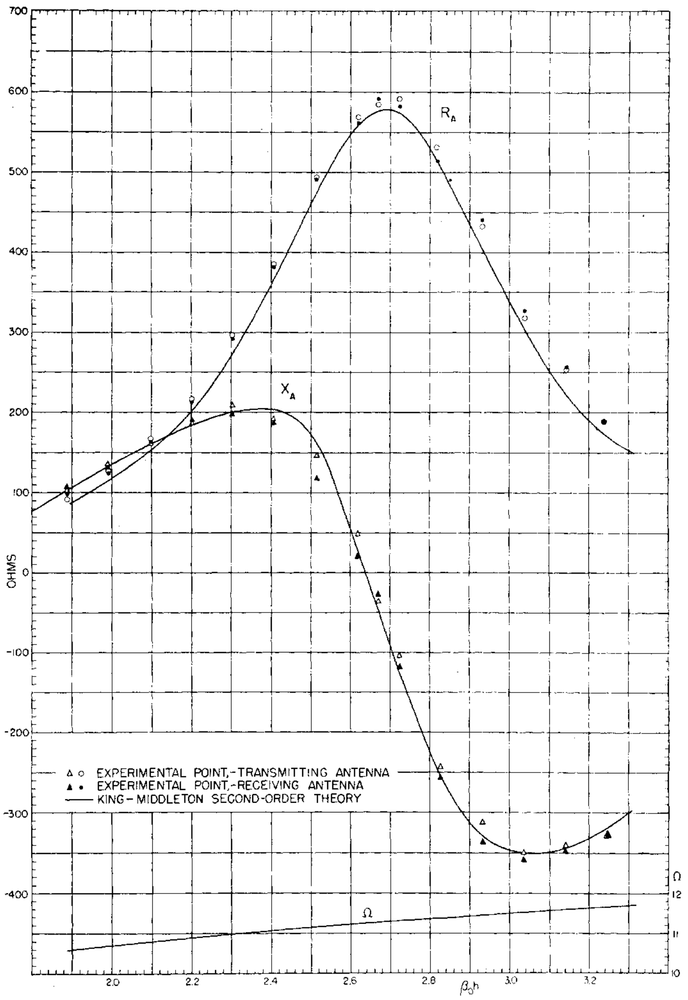

Fig. 5—Theoretical impedance near antiresonance with experimental points from Fig. 3.

The method used to determine ρ_s was to replace the receiving antenna by a metal cap, excite the line by a loosely coupled generator, and measure ρ_s with an auxiliary probe when $\rho_o = 0$. Φ_s, for the piston at the lower end of the line was determined by locating the minimum in the distribution curve using an auxiliary probe.

Although, in general, the measurement of the impedance of an antenna is more convenient when it is driven than when used for reception, since the standing-wave-ratio and distribution-curve methods are available, the receiving-antenna method has the advantage of requiring no transmitter at the location of the measurement. Thus, for example, only light test equipment is needed for measuring the impedance of an antenna in an aircraft or vehicle in motion, using a slotted line with predetermined ρ_s and Φ_s and a signal from a more or less distant high-powered transmitter.

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[6] R. King and D. Middleton, "The cylindrical antenna, theory and experiment," *Quart. Appl. Math.*, vol. 3, pp. 302–335; January, 1946; also *Jour. Appl. Phys.*, vol. 17, pp. 273–284; April, 1946.

[7] D. D. King, "The measured impedance of cylindrical dipoles," *Jour. Appl. Phys.*, vol. 17, pp. 844–852; October, 1946.